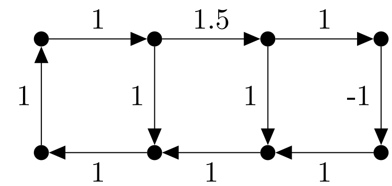
Minimum Mean Cycle

Problem: Given a directed graph $G = (V, E)$ and cost function $c : E \to \mathbb{R}$, find a Minimum Mean Cycle C. That is, minimize $\frac{\sum_{e \in C} c(e)}{|C|}$ over all cycles C. Denote the minimum by $\mu(G, c)$.

1: Find a minimum mean cycle in the following graph.



Solution: If C has mean less than 1, then C contains the right-most edge. By inspection, C is the rectangle's boundary. So $\mu(G, c) = (6 \cdot 1 + 1.5 - 1)/8 = 7.5/8$.

A *walk* is a sequence of alternating vertices and edges $v_1, e_1, v_2, e_2, \ldots, v_k$ where for each i, e_i is from v_i to v_{i+1}. The *length* of a walk is the number of edges in the walk.

Assume there is a vertex s such that every vertex of G is reachable from s.

Let $F_k(x)$ be the minimum cost of a walk from s to x of length k. If no such walk exists, $F_k(x) = \infty$.

2: What happens if there is k such that $F_k(x) = \infty$ for all $x \in V$? If it happens for some k, what is the smallest such k?

Solution: There is no cycle. If there were, we could walk around the cycle infinitely. The smallest k is n. It takes up to $n - 1$ steps to reach all vertices in a directed path.

Let C be the minimum mean cost cycle.

3: Let $x \in C$ and $F_k(x) < \infty$. Compute an upper bound on $F_{k+|C|}(x)$. Find sufficient conditions for $\mu(G, c)$ and $F_k(x)$ to make the upper bound tight.

Solution:

$$F_{k+|C|}(x) \le F_k(x) + \sum_{e \in C} c(e).$$

This is tight when $F_k(x)$ is the least cost walk over all length k walks AND $\mu(G, c) = 0$.

Our goal is to show

$$\mu(G, c) = \min_{x \in V} \max_{\substack{0 \le k \le n-1 \\ F_k(x) < \infty}} \frac{F_n(x) - F_k(x)}{n - k}.$$

4: Assume $\mu(G, c) = 0$. Show that

$$0 = \min_{x \in V} \max_{\substack{0 \le k \le n-1 \\ F_k(x) < \infty}} \frac{F_n(x) - F_k(x)}{n - k}$$

by arguing that $\le$ is always true and that there exists a vertex that has equality.

Solution: $\mu(G, c) = 0$ implies that G has no negative circuit. Hence $F_k(x)$ is $\ge$ the distance from s to x. Hence $\max_k F_n(x) - F_k(x) \ge 0$.
Let C be the minimum mean cost cycle and $w \in C$. Consider a shortest (least cost)

path from s to w followed by n repetitions of C. This has the same cost as the path, so any initial part must be also least cost. Take first n steps of the path and this gives the desired x.

5: Assume $\mu(G, c) = 0$. Let $\delta \in \mathbb{R}$ and let $c' : E \to \mathbb{R}$ be defined as $c'(e) = c(e) + \delta$. (c' is adding δ to the cost for each edge) What is $\mu(G, c')$ and if F' corresponds to c', what is

$$\frac{F'_n(x) - F'_k(x)}{n-k}?$$

Solution: Let C be the minimum mean cycle. Then

$$\mu(G, c') = \mu(G, c) + \frac{\delta|C|}{|C|} = \mu(G, c) + \delta.$$

and

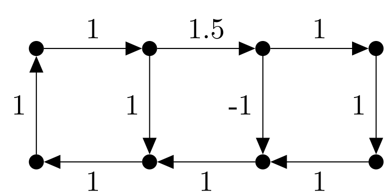
$$\frac{F'_n(x) - F'_k(x)}{n-k} = \frac{F_n(x) + n\delta - F'_k(x) - k\delta}{n-k} = \frac{F_n(x) - F'_k(x)}{n-k} + \delta$$

Since the change is the same for all cycles, we can add δ and the solution would be the same. This leads to the following algorithm.

Algorithm Minimum Mean Cycle:

1. add vertex s and edges sv for all $v \in V$ with $c(sv) = 0$
2. $F_0(s) := 0$; $n := |V \cup \{s\}|$; and $\forall v \in V, F_0(v) = \infty$.
3. for $k \in \{1, \dots n\}$
4. for all $v \in V$
5. $F_k(v) := \infty$
6. for all $\overrightarrow{uv} \in E$
7. if $F_k(v) > F_{k-1}(u) + c(uv)$ then
8. $F_k(v) := F_{k-1}(u) + c(uv)$ and $p_k(v) := u$
9. if $F_n(x) = \infty$ for all $x \in V$, then G is acyclic
10. Find x minimizing $\max_{k, F_k(x) < \infty} \frac{F_n(x) - F_k(x)}{n-k}$.
11. Minimum mean cycle is in $\dots, p_{n-2}(p_{n-1}(p_n(x))), p_{n-1}(p_n(x)), p_n(x), x$

6: Run the algorithm on



Solution: Todo!

7: What is the time complexity?

Solution: $O(mn)$ We need n iterations and in each of them, each of m edges is used once.